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Fourier Optics

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Abstract

Fourier analysis and physical optics are close relatives. On one hand, Fourier analysis is an excellent mathematical tool for the analysis and synthesis of optical systems. On the other hand, optical systems are extremely efficient in performing Fourier analysis of signals and images. Both aspects are reviewed in this paper. For the analysis of optical systems, we invoke a concise notation based on an operator algebra that has its roots in canonical operator theory. After a general discussion, several examples of specific optical systems are described. The second aspect of the relationship between Fourier analysis and optics is demonstrated by several signal-processing architectures that exploit special capabilities of optical systems.

The signal processing applications presented include the optical implementation of a Fourier transformation, one- and two-dimensional spectrum analysis and pattern recognition. Since a conventional Fourier transform processor is space (or time) invariant, space variant signal processing is also addressed.

1. Introduction

This is, by and large, a journal for mathematicians, but the authors of this paper are physicists, so two disciplines meet when readers of this journal examine this paper. Such interdisciplinary encounters can be difficult for all parties but often worth the effort. We hope this interdisciplinary work will be interesting to mathematicians in its own right (the operator treatment) and as a motivation for extending applicable mathematics. One of the major applications of Fourier mathematics is to analyze optics and one of the major applications of optical processing is Fourier analysis of one-dimensional (1D) and two-dimensional (2D) signals with recent extension to three-dimensional signals as well.

Readers interested in current or historical work in Fourier Optics might turn next to *Selected Papers on Fourier Optics* [1], which spans 125 years of technical work in this field. In optics, certain traditional notations exist and will be used here. Input time or space domain signals are represented by lower case functions $f(\vec{x})$, $g(\vec{x})$. Their Fourier transforms are represented by the corresponding capital letters $F(\vec{u})$, $G(\vec{u})$. Complex conjugation is indicated by a *, and $i = \sqrt{-1}$. Finally, we write Fourier transforms in the form

$$F(\vec{u}) = \int_{-\infty}^{\infty} f(\vec{x})\, e^{-2\pi i(\vec{x}\cdot\vec{u})} d\vec{x} \tag{1}$$

and

$$f(\vec{x}) = \int_{-\infty}^{\infty} F(\vec{u})\, e^{2\pi i(\vec{x}\cdot\vec{u})} d\vec{u} \tag{2}$$

All other notation is either explained or transparent.

Fourier methods have been used to describe optics for over 100 years, but only around the 1960's was optics used to describe (or perform) Fourier analysis (O'Neill [2], 1956). The optical generation of Fourier transforms require coherent light and only when lasers were developed in the early 1960's we had sources that were simultaneously bright and highly coherent. It was lasers that stimulated Fourier Optics research, and optical processing was considered seriously

[Cutrona *et al.* [3], (1960), Tsujiuchii [4], (1960)]. Finally, a key piece of the puzzle fell into place with VanderLugt's [5], (1964) introduction of holograms into coherent optical processing. The wealth of concepts developed since then is far more than we can convey here.

In this paper we propose a convenient mathematical approach, the majority of implementational details and subtle nuances. This mathematical approach is detailed in a book (Shamir [6]) and a series of publications to be cited later. We note that our field is also blessed by a classic text *An Introduction to Fourier Optics* (Goodman [7]), which has been updated and revised by its author and is a good reference for newcomers to this field. Several other books have been written, which have strengths different from Goodman's and can also be recommended (Gaskill [8], Yu [9], Stark [10], Papoulis [11])

2. Fundamentals of optical systems

Fourier optics has a set of broad principles and goals. In well over 95% of the work, we assume "coherent light" — light that is monochromatic and originates from a perfect point. Of course, neither condition is physically realizable, but lasers allow satisfactory approximations.

The first step is to place spatial information onto a beam of (coherent) light. One good way to do this is to shine the light through or reflect it from a shaped object. Consider a transparent photographic picture of this page. If we illuminate it with a plane wave (such a wave is described, mathematicaly, by a complex function which is constant over the xy plane when the light propagates in the z direction) larger than the picture, the transmitted light is shaped like this page. Because taking and developing photographs is often too slow or inconvenient, we seek other means, the most common of which is an electronically or optically controllable transparency called an SLM (spatial light modulator). These come in many varieties, modulation mechanisms, modalities, speeds, costs, etc [12].

For our purposes here, these are irrelevant details. Although we assume a transmission SLM, modification of the discussion for reflective SLMs is trivial.

Light propagation is governed by Maxwell's equations, which regard light as an electromagnetic wave. In free space and in most materials, the electric field and direction of propagation are mutually orthogonal. Thus the electric and magnetic fields are transverse to the direction of travel. For the sake of simplicity we assume that the vector nature of the electro-magnetic field can be ignored and we may employ the scalar approximation. We also assume that all fields oscillate with a uniform frequency, ω producing ideal coherence.

A propagating wave is expressed as,

$$e\left(x,y,z,t\right) = a\left(x,y,z\right)e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)} \equiv u\left(x,y,z\right)e^{-i\omega t} \tag{3}$$

where the function u denotes the time independent amplitude and phase of the electric field (the *complex amplitude*) and $\vec{k}$ denotes the wave vector in the local direction of propagation whose magnitude is given by $k = 2\pi/\lambda$. In general, the wave vector can be a complicated finction of position. $\vec{r}$ denotes the usual position vector of a point in 3D space with respect to some origin.

The wave frequency is high ($0.6x10^{15}$ Hz for light in the middle of the visible spectrum) and our electronic detectors are not fast enough to follow the field fluctuations directly. Therefore, we can only observe the total intensity of the wave which is proportional to the time-average of the power incident on a unit area, which, in turn, is proportional to the squared magnitude of the field. Therefore, the observed intensity can be represented by,

$$I = \left|u\left(x,y,z\right)\right|^2 \tag{4}$$

Note that the complex amplitude has lost critical phase information so we cannot describe the flow of light through the system in terms of I and we have to use u. Thus, calculated and measured results are different. A Fourier optical system seeks to produce an output pattern in which all significant information is encoded at the end in intensity and not in phase to make the loss of the phase information upon detection harmless.

An indication of how phase information can be maintained is by taking two waves of the same temporal frequency, denoted by $u_1(x, y, z)$ and $u_2(x, y, z)$, that are present at the same location. The total complex amplitude of the field is now

$$u(x, y, z) = u_1(x, y, z) + u_2(x, y, z) \ . \tag{5}$$

and the intensity given by Eq. (4) has now the form,

$$I = |u_1 + u_2|^2 = |u_1|^2 + |u_2|^2 + u_1 u_2^* + u_1^* u_2 \tag{6}$$

where the variables were suppressed for simplicity. Using the real amplitudes appearing in the definition of Eq. (3) we have,

$$I = a_1^2 + a_2^2 + 2a_1 a_2 \cos[(\vec{k}_2 - \vec{k}_2) \cdot \vec{r}] \tag{7}$$

where it is obvious that the phase information is conserved in the cosine factor.

All representations of light prior to detection are in terms of field vectors. We can represent a possibly complex function *f(x, y)* by transmitting an incident plane wave, $u = 1$, through an SLM of complex amplitude transmission *f(x, y)* positioned in the *xy* plane. This complex field pattern can then be Fourier transformed optically or otherwise operated upon.

For purposes of physical realism, the spatial variation of a is assumed to be much slower than that of the exponential term. Since the factor $e^{-i\omega t}$ appears on all sides of a differential equation, it will be ignored hereafter, and we shall mainly use the *complex amplitude*, *u (x, y, z)*. Scalar diffraction theory [13] is usually evoked to describe the propagation of this complex amplitude through an optical system. A simplified version of diffraction theory is Fourier optics [7], which is the issue of this paper. For purposes of brevity in this paper we employ the shorthand notation of an operator algebra. Although operator algebra has its foundations within rigorous mathematical frameworks [6, 14, 15], we restrict ourselves to a heuristic approach [6, 16], which provides a useful physical insight.

2.1. Free space propagation and operator algebra

Electromagnetic field theory does not provide a basis for the existence of a radiating point source. Nevertheless, for our purpose, we may assume a fictitious point source that radiates electromagnetic energy in a spherically symmetric fashion. A point source of unit magnitude located at the origin would emit a spherical wave with its complex amplitude distribution described by the relation,

$$\mathcal{O}(r) = \frac{1}{i\lambda r} e^{ikr}, \tag{8}$$

A radiating object can be represented as a distribution of an infinite number of point sources. Since Maxwell's equations are linear, the field distribution due to several sources will be a linear superposition of the field from all individual contributions. If the field is generated by a coherent source, we have a linear system that responds to an input by a linear superposition of its components. Viewing free space as a position invariant linear system, the spherical wave of Equation (8) can be interpreted as the impulse response, or point spread function (PSF) of free space. Accordingly, if the source distribution is given by a function $u_{in}(x\ ,y,\ z)$, the complex amplitude distribution at a different location can be evaluated by the convolution integral.

$$u_{out}(x, y, z) = u_{in}(x, y, z) * \mathcal{O}(r) \tag{9}$$

where $*$ denotes the convolution operation. This convolution is, in principle, evaluated in three dimensions. In most optical systems, however, the source distribution is restricted to a plane. The output is then detected over another plane, parallel to and situated at some distance d from the input plane (see Figure 1). It is convenient, therefore, to choose a coordinate system where light propagates mainly in the positive z direction and the xy plane coincides with the input plane. The output plane is also normal to the z-axis at $z = d$. Thus, the convolution integral is carried out over the input plane with x and y as integration variables. The result will be a function over the output plane, also with x and y as variables.

Even for this relatively simple system, the convolution integral of Equation (9) is quite complicated. This, can be substantially simplified if the source and output regions are small compared to the distance d. In this case we may use the *paraxial approximation* that concerns the distance, r which can be written in the form.

$$r = z\sqrt{1 + \frac{x^2 + y^2}{z^2}} \approx z\left(1 + \frac{x^2 + y^2}{2z^2}\right). \tag{10}$$

Whenever the paraxial approximation holds, the PSF of free space can be reduced to

$$\mathcal{O}\left(r\right) = \frac{e^{ikd}}{i\lambda d}\mathcal{Q}\left[\frac{1}{d}\right], \tag{11}$$

where we introduced the quadratic phase factor,

$$\mathcal{Q}\left[\frac{1}{d}\right] \equiv e^{i\frac{k}{2d}\left(x^2+y^2\right)}. \tag{12}$$

The variation of r within the integration region is relatively small, and is approximated in the denominator by d. Using Equation (11) for the PSF, the convolution integral (9) can be rewritten as,

$$u_{out}\left(x, y, d\right) = \frac{e^{ikd}}{i\lambda d}\mathcal{Q}\left[\frac{1}{d}\right] * u_{in}\left(x, y, 0\right), \tag{13}$$

where the convolution is evaluated in two dimensions over the xy plane. Writing explicitly the convolution operation and resubstituting the quadratic phase factor, we obtain

$$u_{out} = \frac{e^{ikd}}{i\lambda d}\int e^{i\frac{k}{2d}\left[\left(x-x'\right)^2+\left(y-y'\right)^2\right]}u_{in}\left(x', y', 0\right)dx'dy'. \tag{14}$$

This integral is generally known as the Fresnel-Kirchhoff diffraction integral, [7, 13] which can be derived in several different ways.

Evaluating the squares in the exponent of (14), rearranging terms and re-substituting the notation for the quadratic phase factor, this expression can be rewritten to yield

$$u_{out}(x, y, d) = \frac{e^{ikd}}{i\lambda d} \mathcal{Q}\left[\frac{1}{d}\right] \int e^{-\frac{ik}{d}[xx'+yy']} \mathcal{Q}\left[\frac{1}{d}\right] u_{in}(x', y', 0)\, dx'dy'. \tag{15}$$

In this expression, the quadratic phase factors are considered as operators in the sense that their variables are to be taken the same as those of the expression on their right. Continuing this line of argument, we observe that the integral is a properly scaled, two-dimensional Fourier transformation (FT). To simplify the notation, we define a generic FT operator by the relation

$$\mathcal{F}[f(x, y)] = \int e^{-2\pi i[xx'+yy']} f(x', y')\, dx'dy', \tag{16}$$

and a scaling operator defined through the relation,

$$\mathcal{V}[a] f(x, y) \equiv f(ax, ay) \tag{17}$$

that applies to any two-dimensional function *f (x, y)*. By definition, each operator is assumed to operate on the entire expression on its right, unless indicated otherwise with the help of brackets.

Substituting the above operators, the diffraction integral can be written in the shorthand form

$$u_{out}(x, y, d) = \frac{e^{ikd}}{i\lambda d} \mathcal{Q}\left[\frac{1}{d}\right] \mathcal{V}\left[\frac{1}{\lambda d}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right] u_{in}(x, y, 0) \tag{18}$$

or

$$u_{out}(x, y, d) = \mathcal{R}[d]\, u_{in}(x, y, 0), \tag{19}$$

where the *free space propagation operator* (FPO) is defined by the relation

$$\mathcal{R}[d] = \frac{e^{ikd}}{i\lambda d} \mathcal{Q}\left[\frac{1}{d}\right] \mathcal{V}\left[\frac{1}{\lambda d}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]. \tag{20}$$

Two other expressions for the FPO can be derived from Equation (13). First, one may directly write

$$\mathcal{R}[d] = \frac{e^{ikd}}{i\lambda d} \mathcal{Q}\left[\frac{1}{d}\right] *, \tag{21}$$

Alternatively, we may operate on the whole expression of Eq. (13) by the FT operator to obtain, employing the convolution theorem [6, 7],

$$\mathcal{F}u_{out}\left(x,y,d\right)=\frac{e^{ikd}}{i\lambda d}\left\{\mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]\right\}\mathcal{F}u_{in}\left(x,y,0\right). \tag{22}$$

Since by Fourier analysis the FT of a quadratic phase factor is also a quadratic phase factor,

$$\mathcal{F}\mathcal{Q}\left[\frac{1}{d}\right]=i\lambda d\mathcal{Q}\left[-\lambda^{2}d\right]\mathcal{F}\ , \tag{23}$$

we obtain from Eq. (22),

$$\mathcal{F}u_{out}\left(x,y,d\right)=e^{-i\lambda d}\mathcal{Q}\left[-\lambda^{2}d\right]\mathcal{F}u_{in}\left(x,y,0\right). \tag{24}$$

An inverse FT on this expression leads to the third form of the FPO:

$$\mathcal{R}\left[d\right]=e^{ikd}\mathcal{F}^{-1}\mathcal{Q}\left[-\lambda^{2}d\right]\mathcal{F}. \tag{25}$$

A complete algebra can be derived for these operators using Fourier analysis [6, 16] or relaying on the more rigorous, group characteristics of these operators [14].

2.2. Thin optical elements

An optical system is comprised of free space and any number of optical elements. In this section, we shall only discuss ideal thin optical elements, for purposes of simplicity. The operation of a thin optical element is determined by its transfer function T defined by the relation

$$u_{out}=Tu_{in}. \tag{26}$$

The operation of a thin lens on an incident wavefront is seldom discussed in elementary optics books. Rather, propagation of light through space is said to combine with the operation of a lens to produce an effect such as imaging. Consider the situation shown in Figure 2, which is described by the most famous formula in optics, the thin lens formula

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{f}, \tag{27}$$

where f denotes the focal length. This does not describe anything but imaging. It does not say what happens at positions in space other than that of the two planes located a distance a and b away from the lens. It certainly does not tell how the lens accomplishes this imaging.

In operator notation, both propagation through space and the thin lens produce spherical wavefronts. An ideal thin spherical lens belongs to the class of phase-only optical elements that introduces an additional phase function without modifying the amplitude distribution

$$E_{out}\left(x,y\right)=E_{in}\left(x,y\right)e^{-i\phi(x,y)}. \tag{28}$$

The transfer function of an ideal lens is of this form and can be written as,

$$\mathcal{T}\equiv\mathcal{L}\left[f\right]=\mathcal{Q}\left[\frac{-1}{f}\right]. \tag{29}$$

where f again denotes the focal length of the lens.

2.3. Basic optical systems

The simplest useful optical system is composed of a thin lens enclosed between two lengths of free space (Figure 3). The input distribution is operated on by an FPO through a distance a. The result is then multiplied by the quadratic phase factor of the lens and then a second FPO operates through a distance b. Thus the whole optical system can be represented by a *transfer operator*, $\mathbf{T}$, given by

$$\mathcal{T}=\mathcal{R}\left[b\right]\mathcal{Q}\left[\frac{-1}{f}\right]\mathcal{R}\left[a\right]. \tag{30}$$

This is a general expression that represents all possible processes that can be performed by a single thin lens. To analyze a specific system, the operators can be manipulated using the operator relations and the specific parameters of the system.

2.3.1. Imaging with a thin lens

Substituting the representation of Equation (20) for the two FPOs in Equation (30) we obtain

$$\mathcal{T} = \frac{e^{-ik(a+b)}}{-\lambda^2 ab} \mathcal{Q}\left[\frac{1}{b}\right] \mathcal{V}\left[\frac{1}{\lambda b}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{b}\right] \mathcal{Q}\left[\frac{1}{-f}\right] \mathcal{Q}\left[\frac{1}{a}\right] \mathcal{V}\left[\frac{1}{\lambda a}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{a}\right], \quad (31)$$

where the constant factors from the two FPOs were combined. Substituting the obvious relation

$$\mathcal{Q}\left[a\right] \mathcal{Q}\left[b\right] = \mathcal{Q}\left[a+b\right]. \quad (32)$$

we obtain

$$\mathcal{T} = \frac{e^{-ik(a+b)}}{-\lambda^2 ab} \mathcal{Q}\left[\frac{1}{b}\right] \mathcal{V}\left[\frac{1}{\lambda b}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{b} - \frac{1}{f} + \frac{1}{a}\right] \mathcal{V}\left[\frac{1}{\lambda a}\right] \mathcal{F}\mathcal{Q}\left[\frac{1}{a}\right], \quad (33)$$

It is easy to see that if Eq. (27) is satisfied, the middle quadratic phase factor reduces to unity since ($\mathbf{Q}\ [0] = 1$). Subsequently, the two FT operations are joined together. Using additional, straightforward relations among the operators results in

$$\mathcal{T} = \frac{a}{b} e^{-ik(a+b)} \mathcal{Q}\left[\frac{1}{b}\left(1 + \frac{a}{b}\right)\right] \mathcal{V}\left[-\frac{a}{b}\right]. \quad (34)$$

Several facts regarding this *imaging system* operator should be observed. First, we see that the input distribution is mapped one-to-one onto the output plane. That is, the object is reconstructed exactly as it was in the input plane except that it has modified scale and orientation. We say that the image is magnified by a factor

$$M = -b/a. \quad (35)$$

There is also a constant factor, a/b, that adjusts the intensity – the power per unit area, such that the power integrated over the whole image is the same as the power integrated over the input plane. Of course, we ignored all losses, such as are caused by the finite size of the system and reflections off lens surfaces. In addition to the above factors, there is a quadratic phase distortion that has positive sign if the distances and focal length are positive. Thus, in a system as discussed here, this phase factor will always be present. Later we shall see that to eliminate the quadratic phase an additional lens is required. Well-known applications of the single lens imaging system are the human eye, the camera, and slide or movie projectors. Such applications involve the detection of power per unit area (irradiance), which is proportional to $|u(x,\ y,\ z)|^2$. Thus, as noted in Section 2, the quadratic phase factor has no direct influence.

2.3.2. Fourier transformation with a thin lens

Returning to the generic transfer operator of Equation (30) we may write the left-hand side FPO in the form of Equation (20) and the right-hand side FPO in the form of Equation (25) to obtain

$$\mathcal{T} = \frac{e^{-ik(a+b)}}{i\lambda b}\mathcal{Q}\left[\frac{1}{b}\right]\mathcal{V}\left[\frac{1}{\lambda b}\right]\mathcal{F}\mathcal{Q}\left[\frac{1}{b}\right]\mathcal{Q}\left[\frac{1}{-f}\right]\mathcal{F}^{-1}\mathcal{Q}\left[-\lambda^2 a\right]\mathcal{F}, \tag{36}$$

If we take $b = f$, the middle **Q** operators are canceled and then the product $\mathbf{F}\mathbf{F}^{-1}$ is also canceled. Commuting now **V** with the **Q** on its right leads to

$$\mathcal{T} = \frac{e^{-ik(f+a)}}{i\lambda f}\mathcal{Q}\left[\frac{1}{f}\left(1-\frac{a}{f}\right)\right]\mathcal{V}\left[\frac{1}{\lambda f}\right]\mathcal{F}, \tag{37}$$

This is a FT with the proper scale and a quadratic phase factor, which can be eliminated by taking $a = b = f$. The physical meaning of this FT is that the light distribution over this plane represents the *spatial* spectral components of the 2-D input signal. Each point over this plane represents a specific 2-D *spatial frequency* in analogy to the temporal frequency of time signals. The intensity distribution over the output plane corresponds to the power spectrum of the input signal.

A straightforward application of such an FT system is a spectrum analyzer. Optical spectrum analyzers are commercially available. These analyzers employ spatial light modulators (SLMs) that are, in principle, programmable electronic transparencies. In the case of temporal signal processing, a signal is displayed as a spatial signal using a one-dimensional SLM called an *acousto-optic modulator* (AOM). In AOMs, temporal signals are converted to ultrasonic waves in a transparent material. From the optical perspective, an acoustic wave is thus transformed into temporally changing density distribution, which modulates the optical wave.

3. Optical systems with more than a single lens

In the previous section, we considered the generic optical system with one thin lens. The basic characteristics of a single lens system are not changed by introducing high quality lenses that compensate for errors introduced by our approximations and other aberrations, the discussion of which is outside the scope of this paper. As already noted above, a wide range of applications is possible for this simple optical system. However, it has some intrinsic deficiencies such as the presence of a quadratic phase factor in the imaging system, image inversion and limited flexibility for advanced applications.

Considering the single lens system as a building block, more complicated and versatile optical systems can be constructed by cascading several of these blocks. It turns out that two lenses are, in principle, adequate to implement most of the so-called all-optical processes.

3.1. Quadratic phase

We discussed the single lens imaging system and noted that a spurious quadratic phase exists that cannot be eliminated. As noted above, this is not a problem in image detection, since the electromagnetic field can only be detected through the transfer of energy from the field to the detecting system (eye, photographic film and photoelectronic detectors). Since this energy is determined only by the squared absolute value of the complex amplitude (square law detection), all phase factors, including the quadratic phase, are eliminated.

The situation is more complicated if the process does not stop at the image plane. We have seen that the transfer function of a thin lens is a quadratic phase factor. Therefore, on the one hand, an image with a quadratic phase factor is similar to the input function in conjunction with a lens. When processing of such an image, this “additional lens” must be taken into account. On the other hand, it is easy to eliminate the quadratic phase by the addition of a compensating lens. For example, in the simple imaging system discussed above, there is a quadratic phase factor with a positive phase. Thus, a converging lens (negative quadratic phase), having the proper focal length, will compensate the phase distortion.

In the case of the FT system, we have seen that the quadratic phase factor can be eliminated by positioning the input function at a distance f from the lens. Such a system may, however, be longer than convenient. To shorten the system one may insert the input directly at the lens and add another lens at the Fourier plane to compensate the quadratic phase distortion. This process is easy to understand if we consider the FPO using the expression of Equation (20). This expression, which describes free space propagation for a distance d, has two quadratic phase factors at its two sides. We may compensate these two phase factors by putting a lens with focal length $f = d$ at the input of this space section as well as at the output (Figure 4). We end up with a FT that is identical, mathematically, with the previous FT, but produced by an optical system having only half that length. Of course, the price is an additional lens.

3.2. Microscope and telescope

In principle, the microscope is a cascade of two imaging systems (the objective lens and the eyepiece). A large magnification can be obtained by multiplying the magnification of these two simple imaging systems, the magnification of which is limited by technological difficulties. Although the quadratic phase factors can be ignored when the image is detected, it is good practice to eliminate them at the various stages of the magnification.

Another way to generate the image of the input distribution is to perform two FT operations in succession. If this is implemented by a cascade of two FT systems, with no quadratic phase distortion, an exact, but inverted image is obtained. The magnification will be unity if two identical FT systems are cascaded. Otherwise, the magnification will be the ratio between the two original magnifications. This is the basic configuration of a telescope.

Imaging by the use of various optical configurations is the most important subject of optics. However in this paper, we are mainly interested in signal processing for which a fundamental modification of the approach is needed. While in the classical applications of imaging the objective is to operate on the input distribution and transform it into another, output distribution, in the case of signal processing, we introduce additional information along the process line. The best known system of this kind is the optical correlator, which is discussed in the next section.

3.3. Spatial Filtering

In the microscopes and telescopes just described, the path between object and final image contains one or more planes where there should be a spatial display of the Fourier transform of the object wavefront. White light, induces a wavelength averaging or blurring of the transform pattern, which is also blurred by the independence (phase incoherence) of light from different parts of the object. With laser (monochromatic and coherent) light, we can essentially eliminate such blurring and achieve a clear physical display of a 2D FT of the 2D input pattern. This allows us to insert a “spatial filter,” which preferentially transmits some regions of the Fourier plane and preferentially attenuates others.

Most applications of Fourier optics involve spatial filtering and subsequent propagation. Holography provides a means of operating on the phase of the Fourier transform as well as its amplitude. VanderLugt [5] formulated the first complex (as opposed to real amplitude masks) spatial filters. To observe the effects of a complex spatial filter, we have to propagate the filtered wavefront, as noted earlier.

4. Optical pattern recognition

By far the most popular application of optical Fourier analysis is pattern recognition and localization. The VanderLugt [5] optical processor, sometimes referred to as the $4f$ correlator, is the "classical" optical correlator.

An input scene is impressed on a laser beam, then is Fourier transformed twice causing the output pattern to be an upside down and backward image of the input: $\mathcal{F}\mathcal{F}f(x,y) = f(-x,-y)$ In between the input and its image planes, in the Fourier transform plane, is found

$$F(u,v) = \mathcal{F}f(x,y). \tag{38}$$

According to the shift theorem,

$$\mathcal{F}f(x-x_o, y-y_o) = e^{-\frac{2\pi i}{\lambda f}(x_o u + y_o v)} F(u,v). \tag{39}$$

If we place a matched filter mask

$$m(u,v) = F^*(u,v) \tag{40}$$

in the Fourier plane and $f\ (x$ - $x_o,\ y$ - $y_o)$ in the input plane, the amplitude of light leaving the Fourier plane is $|F(u,v)|^2\, e^{\frac{-2\pi i(x_o u + y_o v)}{\lambda f}}$. In the output plane we will observe C_{ff} [the autocorrelation function of $f\ (x,\ y)$] centered on (-x_o, -y_o) the image of the centroid of $f\ (x$ - $x_o,\ y$ - $y_o)$. In general, we do not want a matched filter. Often such a filter is not even defined (Caulfield, Haimes and Casasent [17]).

A different approach to optical correlation is the joint transform correlator [18] (JTC). In the JTC the input function, *F*, as well as the reference function, *H*, are placed at the input plane of a single FT system that performs a joint FT of the two functions. The power distribution of this FT is recorded as a transparency (either by film or television camera). This transparency is, in fact, also a kind of hologram with the above-mentioned holographic filter being a special case. It is easy to show that a FT of the power distribution results in several terms, two of which represent the correlation between F and H.

There are two problems in optical pattern recognition that have occupied the researchers' attention for over 30 years since VanderLugt's first paper on optical pattern recognition. First, how do we build the system? Second, how do we design the filter? Within recent years many of us have come to believe that both problems are essentially solved. We will address only the filter design here as it alone deals with Fourier analysis. Before that, however, we must try to answer a more important question: Why should anyone care? After all, FFT chips can perform the same task with far greater accuracy.

Historically, optical and electronic Fourier transforms have developed somewhat in parallel. All through that period the speed and capacity race between them has been close. Here are the advantages of optics. The speed is strictly I/0 (Input/Output) limited. Most input devices, SLMs, and most output devices, usually charge coupled device (CCD) detectors, have been borrowed from television applications and hence have been limited to about 20-30 frames/second. However, recent unpublished work in both America and Russia suggests that 512×512 pixel (picture element) SLMs and CCDs running at 1000 frames/second will soon be widely available. In such cases, one could do one correlation every millisecond with 512×512 pixel I/0 frames. Furthermore, with some optical and mechanical difficulty, one could join four such SLMs and CCDs to produce a 1024×1024 I/0 capability with one millisecond frame time. Thus the I/0 could run at one GBit/sec. The process itself runs at a much faster effective rate. For the moment, at least, optics has a throughput advantage over electronics.

Since the first paper on computer design of pattern recognition filters (Caulfield and Maloney [19]), literally thousands of papers have been published and continue to be published. How can we assert that the problem is solved? It is fairly simple.We design the filter on-line on a SLM [20]. Experimentally, if we want to recognize a set of $N \times N$ pixel objects, we need a $2N \times 2N$ amplitude (real-valued) SLM filter or an $N \times N$ complex SLM filter. More pixels do not help, and fewer pixels do not perform as well, per Nyquist, Shannon, Whittaker, Kotelnikov, etc.

Figure 5 shows a scheme for evolving a filter using genetic algorithms (GA) [21]. Let us choose a figure of merit M that can be measured on such a system. We start with a set of random masks, evaluate their figures of merit on line, select two of them stochastically in some elitist method (preferring good performers), perform crossover, perhaps mutate the offspring, then substitute the new mask into the set replacing two low performers. Eventually a stable result is obtained which (in theory only) is the globally optimum mask. In practice, masks so derived are nearly always excellent, but successively evolved optimum masks may not resemble each other very much in detail or in performance.

Some of the advantages of this method follow. The same method works for any figure of merit. Defects in SLMs, optics, CCDs are automatically taken into account. Masks do not have to be fabricated, inserted, and aligned. The masks so-designed approach and sometimes achieve global optimality.

5. Space Variant Processing

Fourier optics is attractive because it allows space-invariant filtering. Quite early in the development of this field, these methods were extended to space-variant processes [22] [23] [24]. A simple way to analyze space-variant processors is to double the dimensionality of the filter: e.g. a one-dimensional signal with a two-dimensional filter. No general method for designing space-variant filters is offered, but we prefer to use some global optimization method such as genetic algorithms or simulated annealing.

As a “sanity check,” we impose the tracking requirement on this system with the result that the number of free variables is drastically reduced by the needed imposition symmetry. In fact, to duplicate conventional Fourier filtering results we wind up with the same number of free variables as a conventional filter.

To achieve space variant filtering, we need to achieve a vast increase in the number of free variables in our mask. One way to do this is to double the number of filter dimensions: $1D \rightarrow 2D$ or $2D \rightarrow 4D$. For simplicity, we restrict ourselves to 1D in/out with 2D filtering in between.

Consider F and F^{-1} as Fourier transform and inverse Fourier transform operators. They may be lenses, matrices, FFTs, etc., as needed. Likewise we have, for the moment, a 1D input function f_{1D} that could be a function $f\ (x)$, or a vector $\vec{f}_{1D}$, etc., as needed. We will go to a 2D version of f by an operator $D^{\uparrow}$ that increases dimensionality. We can then do 2D Fourier operations on the resulting 2D pattern. Finally, we can use an operator $D^{\downarrow}$ to decrease dimensionality, i.e. reestablish a 1D output g_{1D}. Symbolically

$$g_{1D} = \mathbf{D}^{\downarrow}\mathcal{F}_{2D}^{-1}M_{2D}\mathcal{F}_{2D}\mathbf{D}^{\uparrow}f_{1D}. \tag{41}$$

Here M_{2D} is envisioned as a passive operator [M can stand for <u>M</u>ask or filter <u>M</u>atrix]. Let us view these as optical operators. Then the net 2D operation is

$$\mathcal{O}_{2D} = \mathcal{F}_{2D}^{-1}M_{2D}\mathcal{F}_{2D}. \tag{42}$$

Let

$$f_{2D}^{1} \equiv \mathbf{D}^{\uparrow}f_{1D}. \tag{43}$$

Then

$$\begin{aligned}
\mathcal{O}_{2D}f_{2D}^{1} &= \mathcal{F}_{2D}^{-1}M_{2D}\mathcal{F}_{2D}f_{2D}^{1} \\
&= \mathcal{F}_{2D}^{-1}\left[M_{2D}\mathcal{F}_{2D}f_{2D}^{1}\right] \\
&= \left[\mathcal{F}_{2D}^{-1}M_{2D}\right] * \left[\mathcal{F}_{2D}^{-1}\mathcal{F}_{2D}f_{2D}^{1}\right] \\
&= m_{2D} * f_{2D}^{1}. \qquad (44)
\end{aligned}$$

where

$$m_{2D} = \mathcal{F}^{-1} M_{2D}, \tag{45}$$

and * indicates convolution.

Whatever m_{2D} or M_{2D} we use results in a convolution. So far no special properties of m or M are invoked. We thus have

$$g(x) = \mathbf{D}^{\downarrow}\left[m_{2D}(x, y) * \mathbf{D}^{\uparrow} f(x)\right]. \tag{46}$$

If we consider the special class of transforms,

$$m_{2D}(x, y) = m_{2D}(x - y). \tag{47}$$

and

$$\mathbf{D}^{\uparrow} f(x) = f(x)\, I(y), \tag{48}$$

we obtain

$$g(x) = \mathbf{D}^{\downarrow}\left[m_{2D}(x - y) * f(x)\, I(y)\right], \tag{49}$$

where f(x) I(y) spreads f(x) uniformly in the y direction. Suppose $D^{\downarrow}$ simply represents selection of the y values at x = 0. We should then write

$$g(y) = \left[m_{2D}(x - y) * f(x)\, I(y)\right|_{x=0}, \tag{50}$$

where we have re-introduced the suppressed vertical component to show the dependence on the vertical stack of filters. Also

$$m_{2D}(x - cy) * f(x - x_o) = I(y)\,|_{x=0} = g(y - cx_o), \tag{51}$$

which is a general formula for a “tracking filter.”

When $c = 1$, we have again

$$m_{2D}(x, y) = m_{2D}(x - y), \tag{52}$$

the same as Equation (47). In the discrete case, $m_{2D}(x - y)$ is a Toeplitz matrix.

In general, the m matrix can be space variant. To assure space invariant tracking, we require a circulant Toeplitz matrix or, equivalently, a diagonal M_{2D}. This is the Conventional Spatial Filter or CSF processor. All CSF processors use a diagonal M_{2D}, that is, they multiply each element of F with some value but leave the result where it is. Off diagonal terms involve shifts and adds as well, leading to a more general case that we call a Generalized Spatial Filter or GSF processors.

Various special cases of the GSF lead to new configurations we will not explore here. GSF processors for 1D signals are easy to operate optically. For example

$$\begin{aligned} \mathcal{F}_{2D} f_{2D}^{1} &= \mathcal{F}_{2D} f(x)\, \delta(y) \\ &= \left[\mathcal{F}_{1D} f(x)\right] I(y)\,. \end{aligned} \tag{53}$$

where δ (y) is a delta function, only along the y axis.

Going to 2D inputs and outputs requires 4D transforms. If this is reminiscent of wavelets, it should be. Wavelets are special cases of GSF processors. The most important thing, however, is that the GSF supports space variant filters.

6. Other Applications

Optical Fourier transforms have been applied to a number of applications other than optical system design and pattern recognition. These include image processing: high- and low- pass filtering, morphological transforms, power-spectrum analysis (radio-frequency signals), neural networks, and wavelets.

6.1. High-, Low, and Band-pass Filtering

Spatial filtering applications of the Fourier transform processor were developed to enhance photographs. The optical setup used for high-, low-, and band-pass filtering of spatial frequencies is identical to the conventional 4f system. The filter, usually an amplitude mask, is tailored to transmit the desired spatial frequencies. Examples are given in Refs. [6, 7, 25].

A high-pass filter emphasizes high spatial frequencies. This type of operation can be used to detect edges and corners. An optical implementation would typically consist of an opaque disk covering the center of the Fourier transform plane.

A low-pass filter enhances the low spatial frequencies, yielding a more blurred image, while filtering out high frequency noise. For a two-dimensional input such a filter is typically a circular aperture symmetric about the optical axis.

To obtain a specific range of spatial frequencies, one may construct a band-pass filter. This will pick out a specific range of spatial frequencies.

Other spatial filters constructed for image processing can be used to extract lines of specific orientations, by suppressing spatial frequencies in the FT plane in a line perpendicular to the desired orientation.

6.2. Morphological Transformations

Fourier optical processors are well suited to perform symbolic substitution [26, 27] which is a powerful method for morphological transformations. The optical implementation of the erosion and dialation operations requires a 4f system while opening and closing operations require two 4f systems put back to back.

For the opening and closing operations the first stage of the operation uses a 4f pattern recognizer to find all occurrences of a specific pattern. In the second stage, also a 4f system, a replacement pattern is substituted at the location of each occurrence of the target pattern, by convolving the output of the recognizer stage with the Fourier transform of the desired substitute pattern.

For erosion and dilation, the first 4f stage is not required because the pattern recognition step represents an identity operation. In practice, however, it is simpler to build a single 8f system for all four basic operations. Note that all operations require an optical nonliniarity at the correlation plane, to implement a thresholding operation. This can be achieved with an optical bistability [28].

6.3. Power-Spectrum Analysis

A one-dimensional Fourier transform processor can be used to obtain a radio frequency spectrum. The key to such an instrument is the acousto-optic (AO) modulator [29]. Sound (acoustic) waves are dramatically different from light (electromagnetic waves). Sound is a periodic disturbance of a medium (air, water, metal, etc.). At any location in that medium, the pressure (or concentration) changes periodically. On the other hand, light is an electromagnetic wave not a wave in some physical medium (e.g., the "ether"). Early in this century Michelson and Morley [30] conclusively disproved the existence of the ether and opened the door to relativity theory.

Another profound difference is that acoustic waves in a gas or liquid are parallel to (*longitudinal waves*), not transverse to, the direction of propagation (as is the case with light). Acoustic waves may have also a transverse component when they propagate in solid media. In any case, all waves obey the same basic wave equation and can be treated the same for many purposes (diffraction, interference, holography, etc.).

An AO cell is usually made of glass or crystal. An acoustic wave is induced by piezo-electric transducers located at one side of the cell. The variable material density in traveling acoustic wave forms a diffraction grating within the AO material, which is comprised of local refractive index variations

$$\Delta n\left(z,t\right) = \Delta n \sin\left(\omega_s t - k_s z\right), \tag{54}$$

where ω_s denotes the angular frequency, and k_s denotes the wave vector of the acoustic wave.

When illuminated by a coherent light beam that satisfies the so called Bragg condition,

$$2\lambda_s \sin\theta = \frac{\lambda}{n}, \tag{55}$$

where n is the average index of refraction, the angle θ of the incident light relative to the acoustic wavefronts is a function of the acoustic frequency. Thus light is scattered into directions corresponding to different frequencies.

The output from the AO cell is then Fourier transformed in one dimension to obtain the frequency spectrum. Since the Bragg condition is an acousto-optic version of a thick hologram, however, there is a limit on the maximum allowable bandwidth. Bragg cell spectrum analyzers have been built with bandwidths up to 1 GHz and a time bandwidth product of 1000 [31]

If greater bandwidth is required, one can also use an acousto-optic cell in the Raman-Nath regime. This is an acousto-optic equivalent of a thin phase hologram. The light is scattered into the $\pm$1st order. Raman-Nath is used with thinner cells, and is less efficient in coupling light into these orders.

6.4. Neural Networks

All artificial neural networks share one common characteristic: they require a large amount of interconnections. The 4f optical system provides a simple implementation of space invariant interconnections. Two optical neural networks that make use of Fourier optical systems are due to Soffer, et al. [32], Owechko, et al. [33] and Abu-Mustafa and Psaltis [34].

The system proposed by Soffer's group uses a nonlinear photorefractive crystal to return an amplified phase conjugate of the correlation peak through the 4f system. If this signal now is cycled back through, any cross-correlations will be damped out, eventually yielding the desired output.

Abu-Mustafa and Psaltis' system also cycles the information through the system, but the non-linearity is provided by a pinhole mask in the correlation plane. Amplification for the next cycle is provided by optically addressed spatial light modulator.

Both systems perform well as auto-associative neural networks with noisy and partial inputs. The Abu-Mustafa and Psaltis system can also be reconfigured to function in a hetero-associative mode.

Both of these optical neural network implementations are of the feedback type. These same concepts, however, can be generalized to build multi-layered feedforward systems. In this case the space-invariance of Fourier optical systems is important. It is much more difficult to build space-variant multi-layered neural network system.

6.5. Wavelets

A special issue of Optical Engineering [35] and an invited review in the Proceedings of the IEEE [36] have been devoted to wavelet transforms in optics. Optical Fourier transform processors have been built to obtain wavelet expansions of scenes. These processors, not unlike those used for morphological transforms and neural networks, are variants of the 4f system.

The signal to be wavelet transformed is input into the optical system in one dimension, using an AO cell. It then is Fourier transformed in one dimension, expanded into two dimensions, and subsequently multiplied by a mask representing a mother wavelet and several daughter wavelets. These filters are arranged vertically. The resulting 2D product is now passed through a combination of spherical and cylindrical lenses. The detector plane output is simultaneously Fourier transformed and imaged. The vertical coordinate designates the dilation factor, while the horizontal location designates the location.

This allows "triply continuous" wavelet transforms. A continuous (time-frequency) wavelet transform can be applied to a signal moving continuously through a time window defined by the acoustooptic cell. This, of course, is impossible digitally.

6.6. Fractional Fourier Transforms

Invented by Namias [37] and developed by McBride and Kerr [38] the fractional Fourier transform can also be implemented by optics [39] [40] [41]. Moreover, it has been shown [42] that fractional Fourier transforms are just a subclass of a much wider class of transformations. It is interesting that the nature and the actual result, of relatively complicated mathematical transformations can be partly deduced from the corresponding optical architecture.

Returning to the subclass of fractional Fourier transforms, the basic definition of the α fractional Fourier transform,

$$F^{(\alpha)}\left[\bullet\right] \tag{56}$$

leads to the relations,

$$\begin{aligned} F^{(0)}\left[\bullet\right] &= \left[\bullet\right], \\ F^{(1)}\left[\bullet\right] &= FT\left[\bullet\right], \\ F^{(\alpha)}\left[F^{(\beta)}\left[\bullet\right]\right] &= F^{(\alpha+\beta)}\left[\bullet\right] \end{aligned} \tag{57}$$

for $\alpha \geq 0, \beta \leq 1$.

Like wavelet transforms, Gabor transforms, Wigner transforms, etc. fractional Fourier transforms are mixed time-frequency operators. Except for $\alpha = 1$, the time content is not explicitly represented.

Now consider an optical sequential Fourier transform imaging system. Normally we insert a filter in the $\alpha = 1$ (Fourier transform) plane. This filter is time (or space) invariant. Of course, a filter in the $\alpha = 0$ (image) plane is fully time (or space) variant. If we place the filter between these planes, it is partially space invariant. A more intuitive way of saying this is that there is a neighborhood with diameter roughly d = αA, where A is the input aperture, over which the filtering is essentially space invariant. Of course, d = 0 in the input plane ($\alpha = 0$) and d = A in the Fourier plane ($\alpha = 1$). In between we can place filters to allow local space invariance. This is useful, for example, in fingerprint recognition. Fingerprints are neither random nor static. Fingers are plastic and do not produce the same print on all occasions, but local features stay constant. This suggests $\alpha < 1$ will be superior to $\alpha = 1$ for fingerprint recognition, and our experiments verify this [43].

Recently, we have made digital computations of fractional Fourier transforms easy by showing how to find a matrix $M^{(\alpha)}$ such that

$$\vec{F}^{(\alpha)}\left[\vec{x}\right] = M^{(\alpha)}\vec{x}. \tag{58}$$

Thus we can produce a discrete fractional Fourier transform [44]

6.7. Deconvolution

If our output signal is of the form

$$o = s * b + n, \tag{59}$$

where s is the true signal, b is a blur function, $*$ indicates convolution and n is random noise: we can try to restore s by operating in the Fourier domain, where Equation (59) can be Fourier transformed to produce

$$O = SB + N. \tag{60}$$

Obviously, if we had a B^{-1} filter we could form

$$O' = OB^{-1} = SBB^{-1} + NB^{-1} = S + NB^{-1}. \tag{61}$$

Fourier transforming that would lead to

$$o' = s + n * b^{-1}. \tag{62}$$

That is, we could restore s perfectly except for a problem with the $n * b^{-1}$ term.

Of course, that "small" problem is actually a disaster. Inevitably, b (or B) will have zeros at which points b^{-1} (or B^{-1}) will be infinite. The operation of Equation (61) converts the image to pure noise.

The Weiner filter replaces

$$B^{-1} = \frac{B^*}{|B|^2} \tag{63}$$

with

$$B^{-1} = \frac{B^*}{\left(|B|^2 + |N|^2\right)}. \tag{64}$$

This prevents such catastrophes, but it is not very effective in real applications.

Recently we (Yaroslavsky and Caulfield [45]) showed that with two or more blurs with differing zeros, we can restore images quite satisfactorily. These multiblur filters reduce to the Wiener filter when there is only one blur function. The Wiener filter, in turn, reduces to the inverse filter when $|\mathrm{N}|^2 = 0$.

7. Conclusions

Fourier optics is attractive because of its speed of operation. It is (barely) within the state of the art to perform a sequence of 1024 × 1024 Fourier transforms, masking operations, etc., each millisecond.

The question then becomes: What can we do with all of this high speed processing? We have offered here the preliminary answer the optical computing community had offered, e.g. pattern recognition, and image processing.

Our community has developed a vast array of mathematical tools largely without benefit from Fourier mathematics. We suspect that mathematicians can help us progress further and that we can help you find applications for your work. We invite this interaction.

8. Figure Captions

Figure 1. The coordinate system, showing the input plane and the transform plane

$(x,\ y,\ 0)$, and the translated plane at $(x,\ y,\ d)$.

Figure 2. A thin lens of focal length $f = \left(a^{-1} + b^{-1}\right)^{-1}$

Figure 3. A single lens optical system. The transfer function consists of a free space operator, a quadratic phase operator, and a second free space operator.

Figure 4. A two lens Fourier transform optical system.

Figure 5. The $4f$ optical system used for generating and implementing masks using genetic algorithms.

Figure 1:

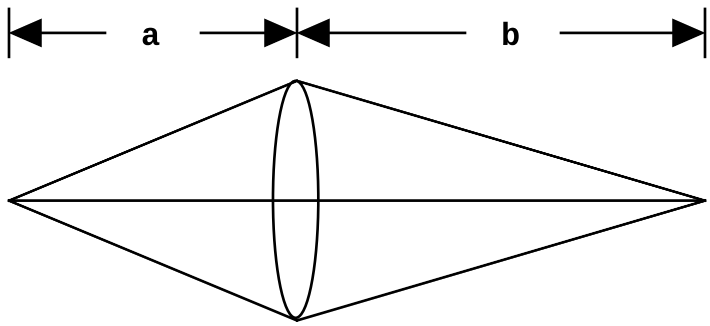

Figure 2:

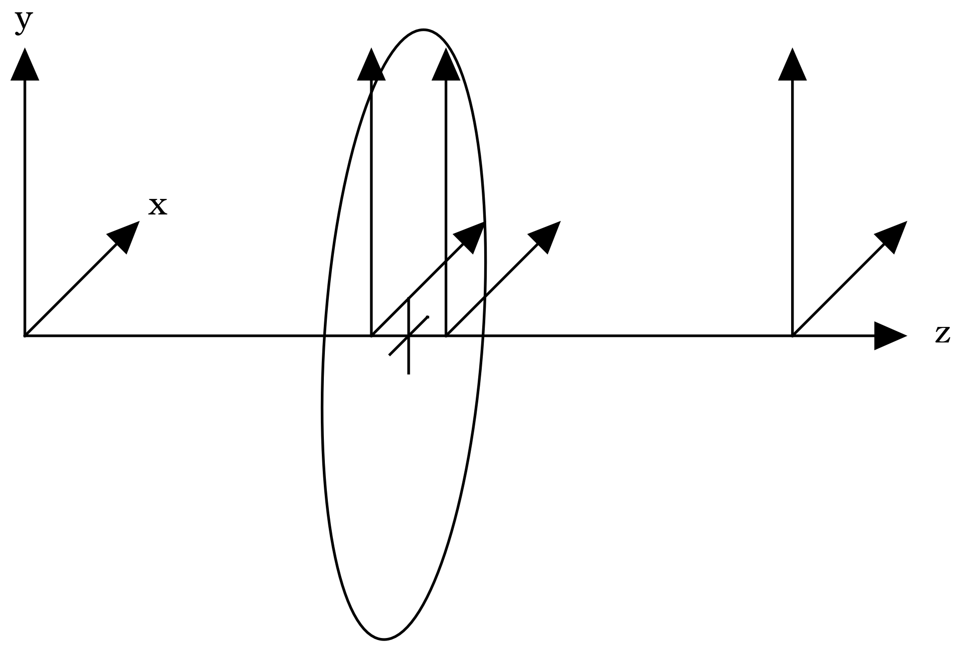

Figure 3:

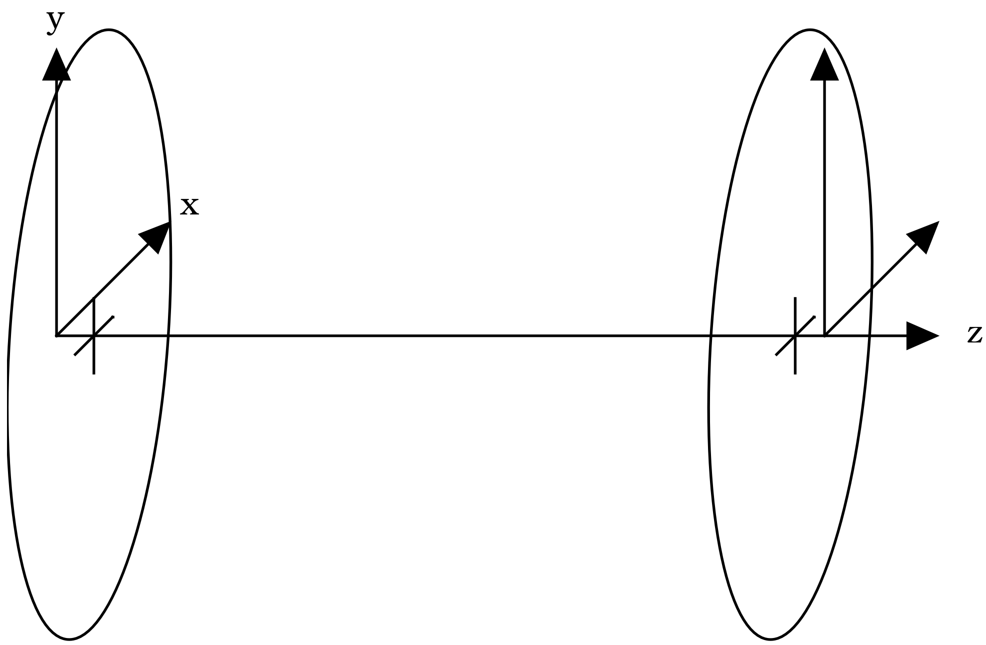

Figure 4:

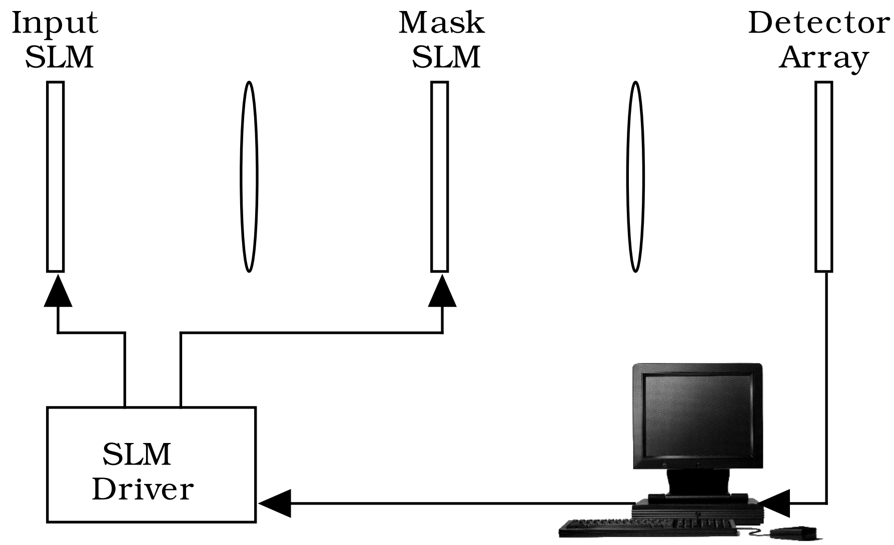

Figure 5:

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Additional Bibliography

One way to come to grasp with the breadth of optical Fourier methods is to look at what work is going on with them now. There follows what we do not pretend to be a complete bibliography of recent papers in this field, but do believe to be representative.

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