Problem Set #2

Date assigned: March 5, 2003
Date due: March 31, 2003

Reading Assignment: Chapter 5 and 6 and your notes on scattering theory.

Problem 1: Let $h_{n_0}(n)$ be the FIR least squares inverse filter of length N with delay n_0 for a sequence $g(n)$, i.e.,

$$h_{n_0}(n) * g(n) \approx \delta(n - n_0)$$

The coefficients $h_{n_0}(n)$ are the solution to the Toeplitz equations (see p. 174 in Chapter 4)

$$\mathbf{R}_g \mathbf{h}_{n_0} = \mathbf{g}_{n_0} \tag{1}$$

which may be solved efficiently using the Levinson recursion. Since the value for the delay n_0 that produces the smallest least squares error is typically unknown, to find the optimum value for n_0 these equations must be solved for each value of n_0, beginning with $n_0 = 0$. Instead of using the Levinson recursion to solve these equations repeatedly, it is possible to take advantage of the relationship between $\mathbf{g}_{n_0}$ and $\mathbf{g}_{n_0+1}$,

$$\mathbf{g}_{n_0} = \begin{bmatrix} g(n_0) \\ g(n_0 - 1) \\ \vdots \\ g(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; ; \; \mathbf{g}_{n_0+1} = \begin{bmatrix} g(n_0 + 1) \\ g(n_0) \\ \vdots \\ g(1) \\ g(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

to derive a recursion for $h_{n_0}(n)$. In this problem we derive this recursion which is known as the *Simpson Sideways Recursion.*

(a) The solution to the normal equations $\mathbf{R}_g \mathbf{h}_{n_0} = \mathbf{g}_{n_0}$ for $n_0 = 0$ may be found using the Levinson-Durbin recursion. Show how to generate the solution for $n_0 = 1$ from the solution for $n_0 = 0$ in less than $4N$ multiplications and divisions where N is the length of the inverse filter $\mathbf{h}_{n_0}$. Note that any information generated in the Levinson-Durbin recursion (for $n_0 = 0$) can be used to construct the new solution.

(b) Generalize the result of part (a) to obtain a recursion that will successively construct the solution for all $n_0 > 0$. Again your method should have less than $4N$ multiplications and divisions at each step.

(c) Write an expression for the error $\mathcal{E}_{n_0}$ at the n_0th step of the recursion in terms of the coefficients $g(n)$ and the coefficients of the least squares inverse filter $h_{n_0}(n)$.

(d) Write a MATLAB program, `sideways.m` that implements the Simpson sideways recursion.

(e) How can this recursion may be used to find the inverse of a Toeplitz matrix?

Problem 2: An important application of least squares inverse filtering is deconvolution, which is concerned with the recovery of a signal $d(n)$ that has been distorted by convolution with $g(n)$

$$x(n) = d(n) * g(n)$$

The problem is to design a filter $h_N(n)$ that will produce an estimate of $d(n)$ from $x(n)$,

$$\widehat{d}(n) = x(n) * h_N(n)$$

One of the problems is that noise in the observed signal may be amplified by the filter. For example, if we observe

$$y(n) = d(n) * g(n) + v(n)$$

then the filtered observations become

$$y(n) * h_N(n) = \widehat{d}(n) + v(n) * h_N(n) = \widehat{d}(n) + u(n)$$

where

$$u(n) = v(n) * h_N(n)$$

is the filtered noise. One way to reduce this noise is to design a least squares inverse filter that minimizes

$$\mathcal{E} = \sum_{n=0}^{\infty} |e(n)|^2 + \lambda E\{|u(n)|^2\}$$

where

$$e(n) = \delta(n - n_0) - h_N(n) * g(n)$$

and $\lambda > 0$ is a parameter that is to be selected. Note that for large values of λ, minimizing $\mathcal{E}$ will force a large reduction in the filtered noise at the expense of a decrease in resolution, i.e., larger $e(n)$, whereas smaller values of λ lead to higher resolution and larger noise.

(a) Assume that $v(n)$ is zero-mean white noise with a variance σ_v^2. Show that

$$E\{|u(n)|^2\} = \sigma_v^2 \; \mathbf{h}_N^H \mathbf{h}_N$$

where $\mathbf{h}_N$ is a vector containing the coefficients of the filter $h_N(n)$.

(b) Derive the normal equations that result from minimizing the error

$$\mathcal{E} = \mathbf{e}^H \mathbf{e} + \lambda \; \sigma_v^2 \; \mathbf{h}_N^H \mathbf{h}_N$$

where $\mathbf{e} = [e(0), \; e(1), \; \ldots]^T$, and show that they may be written in the form

$$(\mathbf{R}_g + \alpha \mathbf{I})\mathbf{h}_N = \mathbf{g}_{n_0}$$

where $\alpha > 0$ is a *prewhitening parameter* that depends upon the values of λ, and $\mathbf{g}_{n_0}$ is the vector on the right-side of Eq. (4.101) in your textbook.

(c) Suppose that we have signal, $d(n)$, that is known to have been blurred by a filter having a unit sample response

$$g(n) = \begin{cases} \cos(0.2[n-25])\exp\{-0.01[n-25]^2\} & ; \ 0 \le n \le 50 \\ 0 & ; \ \text{otherwise} \end{cases}$$

The signal $d(n)$ is a sequence of impulses,

$$d(n) = \sum_{k=1}^{10} d(k)\delta(n - n_k)$$

where the values of $d(k)$ and n_k are as listed in the following table.

n_k	25	40	55	65	85	95	110	130	140	155
$d(k)$	1	0.8	0.7	0.5	0.7	0.2	0.9	0.5	0.6	0.3

Make a plot the observed signal $x(n) = g(n) * d(n)$ and determine how accurately the amplitudes and locations of the impulses in $d(n)$ may be estimated by simply looking at the peaks of $x(n)$.

(d) Using your m-file `sideways.m`, design the least squares inverse filter $h_N(n)$ of length $N = 50$ that has the optimum spiking delay.

(e) Filter $x(n)$ with your optimum spiking filter and plot the output of the filter $\widehat{d}(n) = h_N(n) * x(n)$. What are your estimated values for the amplitudes and locations of the impulses in $d(n)$?

(f) Now suppose that your measurements are noisy, and we have

$$y(n) = g(n) * d(n) + v(n)$$

where $v(n)$ is white Gaussian noise with variance σ_v^2. Filter $y(n)$ using your optimum spiking filter that you found in part (e), and comment on the accuracy of your estimates of $d(k)$ and n_k using $\sigma_v = .1, .01, .001$.

(g) Incorporate a *prewhitening parameter* α in your design of the least squares inverse filter, and modify your m-file `simpson.m` to allow for noise reduction in the least squares inverse filter design. Using this m-file, repeat your experiments in parts (d) and (e) using different values for the prewhitening parameter for $\sigma_v = .1, .01, .001$. Comment of the effectiveness of α in reducing the noise. What values for α seem to work the best? Do you observe any relationship between α and σ_v?

(h) So far, we have assumed perfect knowledge of $g(n)$. Suppose that we design a spiking filter assuming that $g(n)$ is as given in part (c), but in reality, the blurring filter is

$$\widehat{g}(n) = g(n) + w(n)$$

where $w(n)$ is white noise that is uniformly distributed between $[-.005, .005]$. Repeat the design of your least squares inverse filter (without any measurement noise, $\sigma_v = 0$), assuming that the blurring filter is $g(n)$, but using $\widehat{g}(n)$ to generate the measurements

$$x(n) = \widehat{g}(n) * d(n)$$

Filter $x(n)$ with your optimum spiking filter, and plot the output of the filter $\widehat{d}(n) = h_N(n) * x(n)$. How accurate are your estimates of the amplitudes and locations of the impulses in $d(n)$? Could you use a prewhitening parameter to improve your spiking filter design?

Problem 3: Suppose that we would like to study the optical properties of a device consisting of a stack of 3 thin films that have impedances Z_i for $i = 1, 2, 3$. If an electromagnetic wave is normally incident on this layer of thin films, then there will be a series of reflections that take place between the thin films. With the zeroth layer being air, and the third layer having infinite impedance so that it is perfectly reflecting, suppose that the reflection coefficients are

$$c_0 = 0.12, \quad c_1 = 0.15, \quad c_2 = -1$$

where c_0 is the reflection coefficient between air and the first thin film, c_1 is the reflection coefficient between the first and second thin films, and c_2 is the reflection coefficient between the second and last thin film.

(a) Find the reflection response $R(z)$ of this system. Explain your result intuitively.

(b) Draw a lattice filter model for this system of three thin films.

Problem 4:
This problem illustrates the use of inverse scattering theory to design broadband terminations of transmission lines. The termination is constructed by cascading p equal travel-time segments of transmission lines such that the overall reflection response of the structure approximates a desired reflection response. The characteristic impedances of the various elements are obtained from the reflection coefficients $c_0, c_1, \ldots, c_p$. The desired reflection response $R(e^{j\omega})$ is given by

$$R(e^{j\omega}) = \begin{cases} 0 & ; \ 0.2\pi \leq \omega \leq 0.8\pi \\ 0.9\pi & ; \ 0 \leq \omega < 0.2\pi \ \text{ and } \ 0.8\pi < \omega < \pi \end{cases}$$

(a) Using the window design method with a Hamming window, design a $M = 21$ tap FIR filter with unit sample response $r(k)$ with a frequency response that approximates the ideal reflection response $R(e^{j\omega})$ given above.

(b) Write an inverse scattering program in Matlab, `inverse_scatter`, to find the reflection coefficients $c_1, c_2, \ldots, c_p$ corresponding to a given reflection response $r(k)$.

(c) With $p = 6$, use the your inverse scattering Matlab program to find the polynomials $A_p(z)$ and $B_p(z)$, and the reflection coefficients $c_0, c_1, \ldots, c_p$ from the reflection response $r(k)$ that you found in part (a). Plot the magnitude response of the structure, i.e, plot

$$|R(e^{j\omega})| = \frac{|B_p(e^{j\omega})|}{|A_p(e^{j\omega})|}$$

and compare it to the response found in part (a).

(d) Repeat part (c) using $p = 2, 3, 10$.

(e) Repeat parts (a)-(d) using $M = 51$.