TEACHING FOR UNDERSTANDING, NOT JUST ANSWERS

by Sheldon Erickson

As a classroom teacher I feel pressure to cover a large volume of material and I try my best to lead my class through it all. I become frustrated by a lack of mathematical understanding demonstrated by students. It seems they focus on the steps required to get the answer and not on the process of finding solutions. **What is needed to get students past the attitude of simply getting an answer?** I recently had an experience that gave me some insight into this question.

Total Time of Fall (seconds) T	Distance (feet) D
1	16
2	64
3	144
4	256
5	400
6	
7	
8	
T	

Last spring, I was working on a math/science unit integrating forces and motion with proportional reasoning and algebra topics. As a class we had spent a good deal of time during the year looking for patterns numerically and graphically. We had covered square numbers and square roots and had applied them in the study of dimensionality and the Pythagorean theorem. The class had "mastered" the idea of average velocity as a rate and was ready to look at acceleration as a change in velocity.

I chose to give my students some data and let them grapple with making sense of it. To get the class to focus on the situation, I asked the following question: "If you drop a rock off a cliff, do you think there is a way to determine how many feet high the cliff is?" The class agreed that timing the drop was critical but they had no idea how to convert the time to distance. I had copied the AIMS activity page "Have Gravity: Must Travel" (Historical Connections in Mathematics, Volume 1, page 35) and distributed it to my class. They read the description and I made sure they understood that the data were about an object in freefall and the chart listed the total distance the object would fall for differing lengths of time. I then asked them to spend three minutes on their own exploring what patterns they could find. I was amazed at how quiet it became in the classroom. As I made my way around the room, it was clear that there was nearly total engagement. It was obvious that this problem was motivating the students. Were they intrigued by the topic or were they enjoying the mystery of finding patterns? The students were engaged for the three minutes and the only students not working directly on the problem were those that were excitedly sharing patterns they had found. I then asked students to work with a partner and discuss their patterns and the similarities and differences in what they had discovered. At the end of this sharing, I asked students to come up to the board and explain to the class what patterns they had discovered and how they could use their pattern to find the next distance.

Marisha was the first to share. She said she divided the distance by the time and got the following information:

$$\begin{aligned} 16 \div 1 &= 16 \\ 64 \div 2 &= 32 \\ 144 \div 3 &= 48 \\ 256 \div 4 &= 64 \\ 400 \div 5 &= 80 \end{aligned}$$

I asked her why she had done this, thinking she would relate it to her experience of determining speed. She said, "I just thought I'd try something." She did not have a reason. When I pressed the issue of asking her what the quotient of 80 told her about the five-second interval, she thought awhile and responded that it was how far the ball would have to go each second, its speed. When she had done several steps, she realized each quotient was 16 larger than the last one. To get the distance for six seconds she added 16 to the 80 for five seconds giving a total of 96. Then she worked backwards and multiplied 96 by 6 for a product of 576. A number of the students never thought of approaching the problem in this way, but they agreed that they could follow the idea.

Gabe shared that his first thought was distances might be multiples of 16 since the first two examples were. He used a calculator to check out his idea.

$$\begin{aligned} 16 \times 1 &= 16 \\ 16 \times 4 &= 64 \\ 16 \times 9 &= 144 \\ 16 \times 16 &= 256 \\ 16 \times 25 &= 400 \end{aligned}$$

At this point he realized that all the factors used with 16 were perfect squares. To get the next one he would multiply the next perfect square, 36, by 16 to get 576. He went on to add that he noticed that the perfect square was the time multiplied by itself.

Bronson had noticed another pattern as he looked at the numbers. He shared, "All the distances are square numbers." He added another column to his chart.

$$\begin{aligned} 16 &= 4 \times 4 \\ 64 &= 8 \times 8 \\ 144 &= 12 \times 12 \\ 256 &= 16 \times 16 \\ 400 &= 20 \times 20 \end{aligned}$$

He said it was easy to see that the next number would be the product of 24 x 24 or 576.

All the students agreed they had found one of these solution patterns. None of them mentioned the idea of finding finite differences, although I had seen many of them start out using that method. So I suggested that all of us work at the problem from that perspective.

$$\begin{array}{ll} 16 & \\ & > 48 \\ 64 & \\ & > 80 \\ 144 & \\ & > 112 \\ 256 & \\ & > 144 \\ 400 & \end{array}$$

Many in the class admitted they had started this way but were frustrated because no pattern seemed obvious. So I suggested they take a little time to look for patterns. It wasn't 15 seconds until Tay was groaning and waving her hand so hard we could not ignore her. She said the numbers in the second column were always 32 greater than the one before it.

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 16
     >  48
 64          > 32
     >  80
144          > 32
     > 112
256          > 32
     > 144
400
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To see if the rest of the class understood Tay's observation, I asked "How can you use Tay's pattern to find the distance for six seconds?" Most correctly added 32 to 144 for an increase of 176 in the sixth second. One hundred seventy-six added to the 400 feet traveled in five seconds gives a total distance of 576. Tay's pattern worked as well as the other three.

I then wanted to see if the students could generalize their patterns. I had them complete the chart to eight seconds and then I asked them to work by themselves to determine how far something would drop in 50 seconds. I suggested they refer to the pattern they had found or one they felt made the most sense to them. I told them to feel free to use their calculators to get their solutions but to write down the number sentence of what they had entered into the calculator. The class went to work and all had solutions within a few minutes.

Kirk volunteered to show us his method for finding the total dropped distance in 50 seconds. His first step was to multiply 50 by 16. I asked him why he used 16. He said he had followed Marisha's method and the quotient got 16 bigger each second, so he multiplied the number of seconds by 16. His second step was to take the product and multiply it by 50. His number sentence was:

(50 x 16) x 50 = 40,000

To see if the class followed Kirk's explanation, I asked them how the number sentence would change if I changed the time the object dropped. They quickly responded they would just put the new time in instead of 50. This demonstrated they were developing the ideas of variable and equations, so I suggested we make a general number sentence that would tell us how to find the dropped distance for any time. They agreed on:

(T x 16) x T = Drop

Since we had studied different notations for multiplication, some students wanted to shorten it to:

(16T) x T = D

Estarla's first step had been to multiply 50 by 4. When asked why she used four, she said that in Bronson's pattern the two factors were always four times bigger than the time. Then she took this product and multiplied it by itself following the pattern. Her notation was:

(50 x 4) x (50 x 4)
200 x 200 = 40,000

When I asked the class to generalize Estarla's method, they quickly changed it to:

$$(T \times 4) \times (T \times 4) = D$$

I asked them if they could think of a shorter way of doing it. The first suggestion was to shorten the notation to:

$$(4T) \times (4T) = D$$

When it was written this way, Jonathan quickly recognized that the quantities in the parentheses were the same, making this a squaring situation. He suggested:

$$(4T)^2 = D.$$

It took the class some time to make sense of Jonathan's insight and they continued to be more comfortable with the other formats.

Megan said she multiplied 50 by 50 because one factor in Gabe's pattern was the time squared. She multiplied the product by 16 like the pattern. The number sentence was:

$$(50^2) \times 16 = 40{,}000.$$

It was quickly generalized and reformatted by the class to:

$$16(T^2) = D.$$

In closing the lesson that day we wrote down all the generalizations and I asked the class what they had in common.

$$(16T) \times T = (4T) \times (4T) = (4T)^2 = 16(T^2)$$

The consensus was each generalization had either a form of $(T \times T)$ or a form of T^2. The students also noticed there was either a 16 or a four in each generalization and that 16 is the square and four is its root.

As the class left, I had time to reflect. The students had stayed highly engaged for the whole lesson. I felt they had made significant progress and it was a highly successful lesson. What contributed to its success? How could I replicate this success in the future?

My first conclusion was that this **question or context was of interest** to my students. They had an intrinsic curiosity in speed and falling. It was something about which they were curious or at least they saw a purpose in grappling with the situation. The meaningfulness of this context also allowed students to work at a much more abstract level than they normally would. They understood from where the numbers and variables came and saw **equations as descriptions of how they had worked with the context, not as meaningless strings of symbols.** Students continued to work on the problem because the ideas were appropriate for their needs. From my own prior experience, I knew that this context was rich in a variety of math concepts and provided a variety of methods for solutions. **Having learned the prerequisite concepts, my students were free to explore several avenues to discover a solution**.

A second reason I felt this experience was so successful was that **its format presented a patterning situation.** Students seem to have a strong natural affinity for patterns. As I listened to their discussion, I heard their willingness to probe their ideas by testing the reliability of their developing conjectures against the data. They did not want to fail publicly but were willing to explore their ideas when they could confirm their correctness against the data. **As students were able to generalize the pattern, they recognized the power of the understanding they had gained.**

This context's provision of multiple solutions added to the success. When given time, students were willing and even energetic to explore solutions because they were confident it could be done in multiple ways. How much more liberating than to be told there is only one correct answer! The multiple solutions also provided for multiple levels of understanding. While many students could distill their pattern down to a short equation, all students were successful at finding a solution to the question. Jonathan felt proud he had simplified the pattern to $(4T)^2 = D$, while Estarla was so excited to share her solution as (50 x 4) x (50 x 4).

My job as an educator is not to cram as much into a period as possible, but to provide adequate time for students to explore rich situations. It is clear I need to find those inherently interesting contexts, identify the variety of skills and contents that are imbedded in those situations, format the situations so students are free to explore multiple solutions at a variety of levels, and then provide time and guidance for students to clarify and broaden their understanding through discussion and dialogue.

AIMS provides a resource of appropriate activities with skills and content identified. The remaining key element is the teacher who recognizes the opportunity, provides the time, and knows how to craft the right questions.